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Optimal Pointing Control of Robotic Manipulators with State Inequality Constraints

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I. Introduction

THE problem of minimum-time pointing control of a planar, two-link robotic manipulator considers aligning the second link of the manipulator with a remote target point. This problem is quite different from the time-optimal control problems considered in Refs. 1–5 where the final position of the end effector is specified. Pointing a remote target imposes a geometric constraint on both of the joint angles of the manipulator. Therefore, the admissible control space is larger than those considered in Refs. 1–5. In addition, the Lagrange multipliers are also constrained by an equation at final time.

With the additional constraints on the final states and the Lagrange multipliers, optimal solutions have been generated in Ref. 6 for the pointing control problem of robotic manipulators. However, in space applications the second link cannot move freely without any structural interference. Because of this, we consider an inequality constraint to be imposed on the elbow joint angle. It is obvious that the minimum time will be increased. However, it is unclear how optimal solutions will be changed according to the state-variable inequality constraint. For example, both shoulder and elbow joint controls appear linearly in the performance index and the equation of motion; therefore, both controls may be bang-bang or singular depending on two switching functions and their derivatives. It has been shown in Ref. 8 that both controls cannot be singular simultaneously for the problem without state-variable inequality constraint. When the path stays on the state boundary, the two switching functions must be used together with one additional equality constraint to determine both controls. In this Note we will show that even with the state-variable inequality constraint there exists at least one nonsingular control for a rigid n-link robotic manipulator.

The computation of the optimal solutions can be simplified if the constrained and unconstrained arcs can be calculated separately. In Ref. 9 the separate computation has been shown possible for several conditions. We will show that the order of the state-variable inequality constraint for robotic manipulators is two and that the separate computation is impossible for this case. Furthermore, by using the constraint equations, we will show that, if the elbow control is nonsingular, the final point of the optimal path is isolated, if it touches the state boundary.

Received June 26, 1991; presented as Paper 91-2788 at the AIAA Guidance, Navigation, and Control Conference, New Orleans, LA, Aug. 12–14, 1991; revision received Oct. 22, 1992; accepted for publication Nov. 13, 1992.

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II. Problem Formulation

A planar manipulator with two uniform rigid links for pointing control is shown in Fig. 1 where the shoulder joint is fixed to the base; L_1 is the length of the first link, L_2 the length of the second link, r the distance between the center of mass of the second link and the elbow joint, θ_1 the shoulder angle, θ_2 the elbow angle which is the relative angular rotation of the second link with respect to the first link, Ψ the target angle, and R the target distance scaled by L_1. The maximum allowable elbow joint angle is denoted by $\theta_{2\max}$. Equal bounds of magnitude are assumed for the torques T_1 and T_2

$$|T_i| \leq T_{\max}, \qquad i = 1, 2 \tag{1}$$

and the elbow joint angle $\theta_2(t)$ is constrained by

$$|\theta_2(t)| \leq \theta_{2\max} \tag{2}$$

Define the state variables

$$x_1(t) = \theta_1(t), \qquad x_2(t) = \dot{\theta}_1(t)$$
$$x_3(t) = \theta_2(t), \qquad x_4(t) = \dot{\theta}_2(t) \tag{3}$$

and the nondimensionalized control variables

$$u_i(t) = T_i(t)/T_{\max}, \qquad i = 1, 2 \tag{4}$$

with the control bounds set to $+1$ and -1

$$|u_i(t)| \leq 1, \qquad i = 1, 2 \tag{5}$$

We write the dynamic equations as

$$\dot{x}_1 = x_2 \tag{6}$$

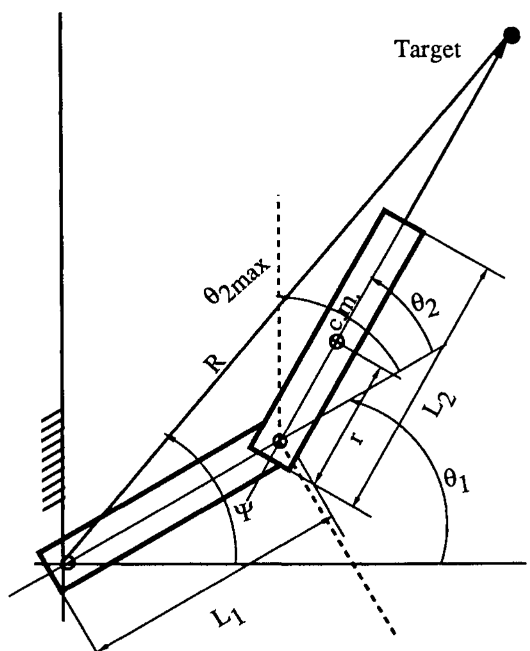

Fig. 1 Two-link robotic manipulator with elbow constraint.

$$\dot{x}_2 = \left[I_2 u_1 - (I_2 + I_4 \cos x_3) u_2 + I_2 I_4 (x_2 + x_4)^2 \times \sin x_3 + I_4^2 x_2^2 \sin x_3 \cos x_3\right] \Big/ \left[I_2 I_3 - I_4^2 \cos^2 x_3\right] \tag{7}$$

$$\dot{x}_3 = x_4 \tag{8}$$

$$\dot{x}_4 = \left[-(I_2 + I_4 \cos x_3) u_1 + (I_2 + I_3 + 2 I_4 \cos x_3) u_2 - I_4 (I_3 + I_4 \cos x_3) x_2^2 \sin x_3 - I_4 (I_2 \cos x_3) \times (x_2 + x_4)^2 \sin x_3\right] \Big/ \left[I_2 I_3 - I_4^2 \cos^2 x_3\right] \tag{9}$$

where I_1 is the mass moment of inertia of the first link with respect to the shoulder axis scaled by $T_{\max}$; I_2 is the mass moment of inertia of the second link with respect to the elbow axis scaled by $T_{\max}$; $I_3 = I_1 + m_2 L_1^2 / T_{\max}$, where m_2 is the mass of the second link; and $I_4 = m_2 r L_1 / T_{\max}$. With the following values selected for the model: $r = L_2/2$, $I_1 = 0.25$, and $R = 100$, the other system parameters are calculated to be $I_2 = 0.25$, $I_3 = 1$, and $I_4 = 0.375$. For the given data, the denominator of Eqs. (7) and (9) are nonsingular for any values of x_3.

For a given target angle Ψ and distance R, the time-optimal control problem is to find the control histories to minimize the maneuvering time required to point the second link to the target

$$J = \int_0^{t_f} \mathrm{d}t = t_f \tag{10}$$

subject to the dynamic constraints (6–9), control constraint (5), state inequality constraint (2), initial conditions

$$x_1(0) = x_2(0) = x_3(0) = x_4(0) = 0 \tag{11}$$

and final conditions

$$x_2(t_f) = x_4(t_f) = 0 \tag{12}$$

The boundary conditions in Eqs. (11) and (12) specify a rest-to-rest maneuvering. If the second link is to follow a moving target point, a different constraint other than Eq. (12) must be added. Here, we assume the target is fixed in space. To align with the target point the joint angles of the robot must satisfy the following constraint:

$$\sin x_3(t_f)/R + \sin\left[\Psi - x_1(t_f) - x_3(t_f)\right] = 0 \tag{13}$$

This geometrical constraint for accurate pointing can be derived using the sine rule and Fig. 1. Since the final shoulder and elbow angles are not prescribed, infinite combinations of $x_1(t_f)$ and $x_3(t_f)$ satisfy Eq. (13). Therefore, the admissible control space is larger than those considered in Refs. 1–5.

III. Necessary Conditions for State-Variable Inequality Constraints

For the unconstrained arc the extremal solution can be determined by using the Euler-Lagrange equation. Define a variational Hamiltonian function

$$H = 1 + \lambda^T(t) f\left[x(t), u(t)\right] \tag{14}$$

where $x(t) = [x_1(t), x_2(t), x_3(t), x_4(t)]^T$ and $u(t) = [u_1(t), u_2(t)]^T$ and where $\lambda(t) = [\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t)]^T$ is the Lagrange multiplier satisfying

$$\dot{\lambda} = -\left[\frac{\partial H}{\partial x}\right]^T \tag{15}$$

The vector function $f[x(t), u(t)]$ in Eq. (14) is the right-hand side of Eqs. (6–9).

Since both controls are linear in the system dynamics, the optimal controls must be determined by Pontryagin's minimum principle which results in two switching functions

$$H_{u_1} = \lambda_2 I_2 - \lambda_4 (I_2 + I_4 \cos x_3) \tag{16}$$

$$H_{u_2} = -\lambda_2 (I_2 + I_4 \cos x_3) + \lambda_4 (I_2 + I_3 + 2 I_4 \cos x_3) \tag{17}$$

for determining $u_1(t)$ and $u_2(t)$. The optimal controls are in their lower bounds of Eq. (5) if the switching functions are greater than zero. On the other hand, if the switching functions are less than zero, the controls with upper bounds optimize the Hamiltonian function. Note that neither of the switching functions are functions of either control. For the cases in which the switching functions are zero for a period of time the singular-arc solutions exist. Singular controls are determined by the second derivatives of the switching functions for the robotic manipulators.[5,7] In addition to the preceding conditions, two boundary conditions must also be satisfied

$$\left[\lambda_1(t_f) \cos x_3(t_f)/R\right] + \left[\lambda_3(t_f) - \lambda_1(t_f)\right] \times \cos\left[\Psi - x_1(t_f) - x_3(t_f)\right] = 0 \tag{18}$$

$$H = 0 \tag{19}$$

Note that Eq. (18) exists only for the case when the final state is not on the state boundary.

If the extremal solution is on the state constraint boundary, we have the following interior boundary conditions:

$$S\left[x_3(t)\right] = x_3^2 - \theta_{2\max}^2 = 0 \tag{20}$$

$$\dot{S}\left[x_3(t)\right] = 2 x_3(t) x_4(t) = 0 \tag{21}$$

$$\ddot{S}\left[x_3(t)\right] = 2 x_4^2(t) + 2 x_3(t) \dot{x}_4(t) = 0 \tag{22}$$

When only Eqs. (20) and (21) are equal to zero, the extremal path touches the boundary at one point. If a constrained arc exists, Eqs. (20–22) all must be satisfied. Since control variables appear in Eq. (22), we can form a new Hamiltonian function as

$$\bar{H} = 1 + \lambda^T f + \mu_1 \ddot{S} \tag{23}$$

where the influence function $\mu_1(t) \geq 0$ on the constrained arc.[10] The Euler-Lagrange equations are still applicable to the new Hamiltonian function $\bar{H}$. However, the Lagrange multipliers exhibit jumps at the entering point of the constrained arc, and the jumps are not unique if the Lagrange multipliers are discontinuous at the exiting point.[10] Here we assume that the Lagrange multipliers are continuous at the exiting points. The additional unknown jumps at the entering points and $\mu_1(t)$ are determined by including the constraint equations [Eqs. (20–22)] with the Euler-Lagrange equations to form a multiple-point boundary-value problem. However, the controls need to be determined by using Eqs. (16) and (17) together with Eqs. (22) and (9).

IV. Structure of Time-Optimal Controls with State-Variable Inequality Constraint

An interesting observation about the final point of the path can be seen from Eq. (22). Assuming the extremal path exits at the final point of the constrained path, we have the following equations:

$$x_4(t_f) = 0, \qquad \dot{x}_4(t_f) = 0 \tag{24}$$

By using the final state boundary conditions given by Eq. (12), Eq. (22) becomes

$$-(I_2 + I_4 \cos \theta_{2\max}) u_1 + (I_2 + I_3 + 2 I_4 \cos \theta_{2\max}) u_2 = 0 \tag{25}$$

For $u_2 = +1$, Eq. (25) is reduced to

$$\cos\theta_{2\max} = \frac{I_2 u_1 - I_2 - I_3}{I_4(2-u_1)} \tag{26}$$

If the fact that $\cos\theta_{2\max}$ is less than $+1$ is used, u_1 is solved from Eq. (26) using the value for I_2, I_3, and I_4 as

$$u_1 \leq -4 \tag{27}$$

If $u_2 = -1$ is assumed, Eq. (25) becomes

$$\cos\theta_{2\max} = \frac{I_2 u_1 + I_2 + I_3}{I_4(-2-\mu_1)} \tag{28}$$

Using $\cos\theta_{2\max} \leq 1$, we obtain

$$u_1 \geq 4 \tag{29}$$

However, the results of Eqs. (27) and (29) contradict the control inequality constraint $|u_1| \leq 1$. Therefore, we have Proposition 1.

Proposition 1: If the final state $x(t_f)$ of the extremal path for the model considered in Sec. II is on the constrained state boundary, then $x(t_f - \epsilon)$, where $\epsilon \to 0$ and $\epsilon > 0$, is not on the constrained state boundary if the elbow control is assumed to be nonsingular.

Several authors (see Ref. 8) have investigated the structure of time-optimal controls for robotic manipulators and have shown that all the controls of a rigid n-link manipulator cannot be singular simultaneously. With the state-variable inequality constraint we also show that there exists at least one nonsingular control.

Proposition 2: The time-optimal controls of a rigid n-link robotic manipulator with state-variable inequality constraints on joint angles contain at least one nonsingular control and the order of the state-variable inequality constraint is equal to two.

Proof: The equation of motion of a rigid n-link robotic manipulator can be written in the form

$$\dot{v} = M^{-1}(q)\left[u - C(q,v) - G(q)\right] \tag{30}$$

$$\dot{q} = v \tag{31}$$

where q denotes the generalized coordinate vector, v the generalized velocity vector, M the inertia matrix, u the control vector, C the Coriolis and centrifugal force vector, and G the gravitational force vectors. Note that the G term is not considered for the two-link manipulator model in Sec. II and M is a nonsingular matrix. The state-variable inequality constraint is given as

$$S(q) = Kq \leq Kq_{\max} \tag{32}$$

where K is an $r \times n$ matrix and where $r < n$. Derivatives of Eq. (32) give

$$\dot{S}(q,v) = Kv \leq 0 \tag{33}$$

$$\ddot{S}(q,v,u) = K\dot{v} \leq 0 \tag{34}$$

From the preceding equations, we prove that the order of the state-variable inequality constraint is always two if the constraint is only on the joint angles.

Form a variation Hamiltonian function

$$\begin{aligned}\bar{H} &= 1 + \lambda_v^T M^{-1}(q)\left[u - C(q,v) - G(q)\right] + \lambda_q^T v + \mu^T K\dot{v} \\ &= 1 + (\lambda_v^T + \mu^T K) M^{-1}(q)\left[u - C(q,v) - G(q)\right] + \lambda_q^T v \end{aligned} \tag{35}$$

and write the switching functions as

$$\bar{H}_u^T = M^{-1}(q)[\lambda_v + K^T\mu] \tag{36}$$

Therefore, if all the controls are singular,

$$\lambda_v + K^T\mu = 0, \qquad \forall t \in [t_1, t_2] \tag{37}$$

$$\dot{\lambda}_v + K^T\dot{\mu} = 0, \qquad \forall t \in [t_1, t_2] \tag{38}$$

Furthermore,

$$\dot{\lambda}_v = -\frac{\partial \bar{H}^T}{\partial v} = \frac{\partial C^T}{\partial v} M^{-1}(q)[\lambda_v + K^T\mu] - \lambda_q \tag{39}$$

Combining Eqs. (37–39) we arrive at

$$\lambda_q = K^T\dot{\mu} \tag{40}$$

Therefore,

$$\lambda_q^T v = \dot{\mu}^T K v = 0 \qquad \forall t \in [t_1, t_2] \tag{41}$$

since $Kv = 0$ on the state boundary. However, Eqs. (37) and (41) contradict the fact that $\bar{H}$ must be equal to zero for time-optimal control problems with time not appearing explicitly in the system. This completes the proof for Proposition 2.

From Proposition 2 we know that one of the elbow or shoulder controls for the two-link manipulator must be nonsingular. The general pattern of the minimum-time solutions in Ref. 6 also shows that the elbow always uses bang-bang control, and the shoulder is only considered as a balance control to the elbow. Furthermore, Proposition 1 implies that if the elbow control is nonsingular, then the second link will not stay on the state boundary all the way to the final time. If the elbow reaches the state boundary, then it must leave the boundary before the final time.

V. Numerical Solutions

It is easier to obtain the extremal solutions that satisfy the necessary conditions if the constrained and unconstrained paths can be calculated separately. Some conditions for the separate computation of the arcs have been developed in Ref. 9. The conditions state that the order of the state-variable inequality constraint (denoted by p) must be equal to the order of the dynamics (denoted by m) or $p = m - 1$ for which the dynamics and boundary conditions do not depend on clock time. As shown in Proposition 2, for the time-optimal control problem of robotic manipulators with state-variable inequality constraint, $p = 2$ and $m = 2n$. Thus, $\min(m - p) = 2(n-1) \geq 2$ for $n \geq 2$, where n is the number of the links. Therefore, the separate computation of constrained and unconstrained arc is not possible for a robotic manipulator with at least two links and with constraints on the joint angles.

Numerical solutions for which the optimal path touches the state boundary only at the final point have been determined by using the minimizing-boundary-condition method.[6,7] The minimizing-boundary-condition method is an alternative method of multiple-point shooting methods. The minimizing-boundary-condition method uses nonlinear optimization techniques to solve two-point boundary-value problems in a way similar to the shooting methods. Unlike the shooting methods, it is not sensitive to the first initial guess. Multiple-point shooting methods have been shown to improve the initial guess sensitivities; however, multiple-point shooting methods may lose accuracy in each node point. The minimizing-boundary-condition method has been used successfully for periodic optimal control problems that require a very accurate algorithm since the minimizing periodic solution is extremely close to the steady-state solution and the performance index is very flat.[6] Furthermore, it is easy to write a computer program for the minimizing-boundary-condition method since a large number of existing nonlinear optimization programs can be used.

Figure 2 gives the optimal path and controls for $\Psi = 91.7$ deg with elbow joint angle limited by ± 109.4 deg. The minimum time increases 8.2%. The elbow control is positive from $t = 0$ to $t = 4.33$ and negative from $t = 4.33$ to $t = 1$. Figure 3 shows the optimal path and controls for $\Psi = 180$ deg with the elbow joint angle limited by ± 175.9 deg. The minimum time increases 6.2% in this case. The elbow control is positive from $t = 0$ to $t = 4.22$ and negative from $t = 4.22$ to $t = 1$. The paths touch the constraint only at the final time and both elbow and shoulder controls are nonsingular. However, the results do not exclude the possibilities for the final point to be a part of a constrained arc or the constrained arc to appear before the final time. As seen from the plots, the shoulder controls with state constraint are quite different from the shoulder controls without state constraint. On the other hand, the elbow controls are similar.

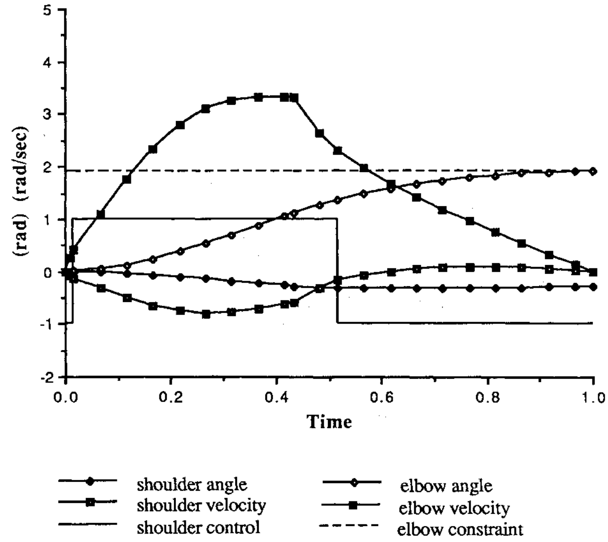

Fig. 2 Solutions for $\Psi = 91.7$ deg with state constraint.

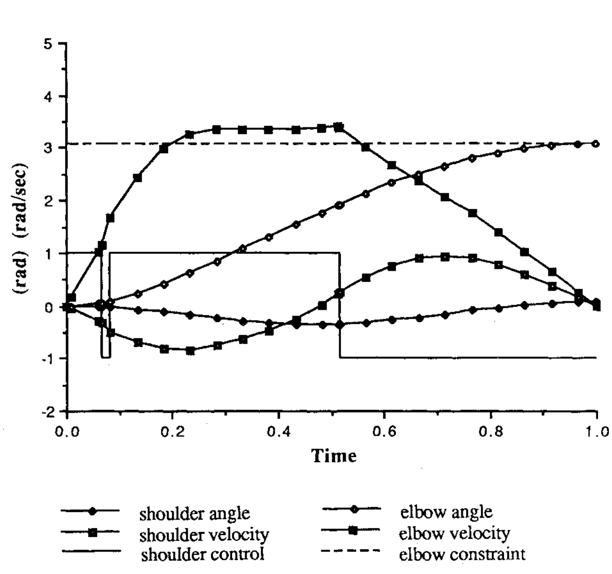

Fig. 3 Solutions for $\Psi = 180$ deg with state constraint.

It should be noted that the minimum time for the robotic manipulator with its shoulder fixed is $\sqrt{\Psi}$. If the constraint angle is small enough, then the model with fixed shoulder link may perform better than the model with free shoulder link. However, the shoulder control will be greater than the given bound. Therefore, under the assumptions that both the shoulder and elbow controls have the same capability, the model shown in Fig. 1 is still better. Note that in Fig. 3 both shoulder and elbow controls are very similar except for the additional switching of the shoulder control at 6.74% of the final time. Also, both controls have a switching at 50% of the final time. This suggests that the controls may approach the controls for two coupled double integrators.

VI. Conclusions

For practical applications, the joint angles of robotic manipulators or of multibody space systems are limited by the physical structure. In this Note numerical methods and structure of time-optimal control related to this state-variable inequality constraint have been developed. By considering the additional jumps of the Lagrange multipliers at the entering points, the minimizing-boundary-condition method has been extended to solve the multiple-point boundary-value problem. The numerical results show some optimal paths which touch the constraint boundary only at the final point of the path. The existence of such paths is predicted analytically using the constraint equations. However, the other possibilities are not excluded. We have shown that the time-optimal control contains at least one nonsingular control even with the path on the state boundary. The order of the state-variable inequality constraint in the joint angles is always two for robotic manipulators, and the separate computation of constrained and unconstrained arcs is not possible.

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